J Popul Econ (2013) 26:907–941
DOI 10.1007/s00148-012-0445-5

ORIGINAL PAPER

Differential fertility and intergenerational mobility under private versus public education

C. Simon Fan · Jie Zhang

Received: 22 April 2010 / Accepted: 5 September 2012 /
Published online: 28 September 2012

Abstract We study differential fertility and intergenerational mobility in an overlapping-generations framework with skilled and unskilled individuals. Assuming unskilled parents are less productive in educating children, we show that they choose higher fertility but less investment for child education than skilled parents. Public education reduces the fertility gap but may increase intergenerational mobility under certain conditions. We also find very different responses of fertility differential and intergenerational mobility to a variation in a preference or technology parameter. As the ratio of skilled to working population rises towards its steady state, average income rises, average fertility falls, but income inequality first rises and then falls.

Keywords Differential fertility · Intergenerational mobility · Education · Inequality

JEL Classification D1 · D3 · I2 · H2 · J1 · O1

Responsible editor: Junsen Zhang

Electronic supplementary material The online version of this article (doi:10.1007/s00148-012-0445-5) contains supplementary material, which is available to authorized users.

C. S. Fan
Department of Economics, Lingnan University, Tuen Mun, Hong Kong

J. Zhang (✉)
Department of Economics, National University of Singapore, Singapore, 117570, Singapore
e-mail: ecszj@nus.edu.sg

J. Zhang
Chongqing University, Chongqing, China

1 Introduction

Intergenerational earnings mobility, together with income inequality, is an important topic of research in the economics literature because it relates to both intergenerational efficiency and intergenerational equity, which is a fundamental concern for economists and policymakers. Indeed, as noted by Grawe and Mulligan (2002), the study of intergenerational correlations of economic and social status is one of the oldest research agendas in the social sciences discipline. In fact, there is a large literature that studies intergenerational transmission of human capital and earnings.[1] The existing models of intergenerational mobility and income inequality, however, have mostly abstracted from the consideration of endogenous fertility. While this omission is largely innocuous for the main purposes of the existing literature with tractability, assuming endogenous fertility along with human capital investment for children can enhance the understanding of how parental factors influence children's earnings, given the typical trade-off between the number and the quality of children in the spirit of Becker and Lewis (1973) and more recently de la Croix and Doepke (2003, 2004) and Moav (2005) for differential fertility. Different fertility rates between agents in different income groups can help to reveal different allocations of time to working and educating children as well as different allocations of income to consumption and investment in child education. The resultant evolution of income distribution and population dynamics are expected to differ from those in existing models without differential fertility.

This paper studies intergenerational mobility in a model that incorporates endogenous and differential fertility with heterogeneous agents differentiated by their skills. Different from our approach, the existing literature on intergenerational earnings mobility abstracts from fertility differences across individuals, while the existing models on differences in fertility across income/skill groups abstract from the analysis of intergenerational mobility. This paper argues that some potentially important insights regarding the evolution of the wealth distribution and intergenerational mobility can be obtained by introducing differential fertility and intergenerational mobility in the same model. Mainly, once faced with a trade-off between the quality and quantity of children, low-skilled parents might choose to have more children and invest less in the education of each child, and thus upward mobility becomes harder to achieve. Moreover, the fact that population growth at the lower end of the distribution is faster has implications for the evolution of the wage distribution in society.

[1] A very partial list of this literature includes, for example, Becker and Tomes (1979), Loury (1981), Piketty (1995), Galor and Tsiddon (1997), Owen and Weil (1998), Iyigun (1999), Maoz and Moav (1999), Mookherjee and Ray (2003), Davies et al. (2005), Cremer and Pestieau (2006), and Docquier et al. (2007). Moreover, see Piketty (2000) for a survey of this literature.

Under the consideration of differential fertility, the educational resources and education outcome for the children with skilled and unskilled parents will be more uneven than the case under the implicit assumption of equal fertility. Thus, income redistribution from the rich to the poor may be particularly important in equalizing educational resources, which may increase intergenerational mobility, efficiency, and equity. Specifically, this paper considers one way of income redistribution that completely eliminates the inequality of educational resources: public education. Thus, this paper also adds to the literature on the relative merits of public education versus private education.[2] In doing so, we extend the analysis of differential fertility in de la Croix and Doepke (2003, 2004) and Moav (2005) by considering intergenerational mobility. In their models, there is no intergenerational mobility because poor families make less or no investment in children's education and therefore suffer persistent poverty. We permit both downward and upward mobility in the education process. Also, we extend the analysis of intergenerational mobility in Davies et al. (2005) and many others that assume fixed fertility by endogenizing fertility and considering differential fertility.

Our analysis is based on a unified framework of population dynamics and income dynamics with heterogeneous agents. We assume that there are two types of individuals: skilled and unskilled. We analyze the evolution of the ratio of skilled to working population over time. Our basic analytical framework extends Becker and Lewis (1973) by postulating that the marginal contribution to children's education of unskilled (skilled) workers' time beyond a necessary level diminishes to zero (diminishes but always remains positive). This postulation yields empirically plausible implications: unskilled workers choose higher fertility but spend less time and income on a child's education than skilled workers. In particular, the positive relation between parental time for child education and parental skills is consistent with empirical evidence in Behrman et al. (1999), Lynch (2000), and Ramey and Ramey (2010). Also, we allow for upward and downward intergenerational mobility in education achievement and earnings by assuming that children from skilled (unskilled) parents have a positive probability to become unskilled (skilled). Through incorporating population dynamics into the basic analytical framework, we investigate the properties and implications of the model for intergenerational mobility under public education and private education, respectively.

The current paper also yields several implications for development and for government policies. First, it examines the relationship between income inequality, fertility, and economic development. The development process in our model is driven by the rising ratio of the skilled to working population that converges to a steady-state value. In an early stage of development that lacks skilled labor, an increase in the ratio of the skilled to working population

[2]This literature includes, among others, Glomm and Ravikumar (1992), Zhang (1996), Epple and Romano (1998), Hassler and Mora (2000), Fernandez and Rogerson (2003), de la Croix and Doepke (2004), Davies et al. (2005), and de la Croix and Doepke (2009).

will increase both average income and income inequality. However, in a later stage of development with more skilled than unskilled labor, a further rise in the ratio of the skilled to working population will increase average income but reduce income inequality. The result resembles the Kuznets hypothesis from a different angle, which complements the existing literature. Moreover, as skilled workers have fewer children than unskilled workers in our model, average fertility is high in the early stage and falls to a lower long-run level in the development process. Thus, it depicts a pattern of a rising trend in average income and a falling trend in average fertility, which is consistent with their observed trends in the last two centuries. Without differential fertility, this falling trend in fertility would not arise in this model.

Second, under some reasonable conditions, it shows that switching from private to public education reduces the relative gap between fertility rates of unskilled and skilled workers. This result is similar to that in de la Croix and Doepke (2004). However, with public education, inequality vanishes in the long run in their model without upward and downward intergenerational mobility. By contrast, inequality remains in the long run in our model in the regime of public education, because the mobility exists in our model. Third, it demonstrates that if the wage differential between skilled and unskilled workers is sufficiently large or if the taste for child education is sufficiently strong, then intergenerational mobility is greater in the public education regime than in the private education regime. This result is obtained partly because the impact of parental income differential on children's education outcome is prevalent in the private education system but it is eliminated by public education, as in the literature. Also, public education raises the downward mobility of children with skilled parents by reducing the amount of time skilled parents spend on their children's education.

However, the new light this paper sheds on intergenerational mobility comes from the negative response of the relative fertility differential between unskilled and skilled workers to public education. This reduced relative fertility differential under public education means more children from skilled workers compared to the private education regime and tends to raise the ratio of skilled to working population in future times, other things being equal. This rise in the skilled portion of the future population can in turn raise average income and hence government spending on education per child in future times (particularly so if the wage differential is large). Subsequently, the rise in future government spending under public education will brighten the chance for children with unskilled parents to become skilled (upward mobility) and will also mitigate the increased downward mobility of children with skilled parents. As a result, switching from private to public education tends to raise the downward mobility of children with skilled parents more in the short run than in the long run on the one hand and raise the upward mobility of children with unskilled parents in all times on the other hand.

Moreover, we find very different responses of fertility differential and intergenerational mobility to a variation in a preference or technology parameter. Finally, in the steady state, whether or not the public education system yields a higher level of average income than private education depends on the amount of public expenditure on education.

The rest of the paper proceeds as follows: Section 2 introduces an extended Becker–Lewis model. Section 3 describes the population dynamics. Sections 4 and 5 derive household decisions under private and public education, respectively. Section 6 examines the determinants of intergenerational mobility under private and public educational systems. Section 7 analyzes the properties of the steady states of the dynamic systems. Section 8 concludes.

2 A simple model of fertility and education investment

This section is an extension of a Becker–Lewis type model.[3] Consider an economy with an infinite number of overlapping generations. Each working generation has a mass N_t and each individual lives for two periods: childhood and adulthood (parenthood). An individual makes decisions only in the second period of life, choosing how many children to bear and how much education investment for each child in addition to his own consumption. When children mature and have had their skills formed, they in turn choose how many children to have and how much education to give to their children. There are two types of parents in the economy: skilled parents—individuals who, for example, receive higher education, and unskilled parents. The preferences of both skilled and unskilled parents are identical as given below:

$$U(n, e, c) \equiv \ln n + \alpha \ln e + \beta \ln c \tag{1}$$

where α and β are positive coefficients indicating the tastes for child education outcome e and for own consumption c relative to the taste for the number of children n (unity). The simple form of the utility function helps to keep the model tractable in dealing with differential fertility, income inequality, and intergenerational earnings mobility.

A parent is endowed with one unit of time, which is devoted to working, rearing children, and educating children. In line with the existing literature, we assume that taking care of children, both quantity-wise and quality-wise, is time intensive for parents.[4] Indeed, many empirical studies show that parental time accounts for the bulk of the cost of producing and rearing children (see, for example, the survey by Becker 1991). Let ξ denote the exogenous time

[3]There is already a large literature on "endogenous fertility," which explains the negative correlation between parental education (or income) and fertility from different angles. The model developed in this section is most closely related to and different from de la Croix and Doepke (2004). Our model implies a negative correlation between parental education and fertility under both private and public educational systems, while their model implies the same fertility for all individuals, skilled or unskilled alike, under public education in which everyone receives the same amount of educational expenditure.

[4]For example, see Galor and Weil (2000), Galor and Galor and Moav (2002, 2004, 2006), de la Croix and Doepke (2003), and Varvarigos and Zakaria (2012).

cost of rearing a child. The time input for child education in fact comprises two components: one of them can directly enhance children's learning (teaching at home in addition to school education) and the other cannot directly enhance children's learning but it provides necessary conditions for children's learning (e.g., monitoring, encouragement, and logistic support). The former input requires parents to possess skills, while the latter input represents a certain minimum level of time committed by parents to the education of each child, denoted by $\tilde{k} > 0$. We thus assume the following:

Assumption 1 *Parental time for a child's education has a lower bound $k \geq \tilde{k} > 0$. Also, the marginal contribution to children's education of skilled parents' time input is always positive, whereas the marginal contribution to children's education of unskilled parents' time input diminishes to zero beyond the necessary level $\tilde{k}$.*

The assumption hinges on the very intuition that the lack of mental labor or skills makes additional parental time inputs helpless to the education of children.[5] With Assumption 1, a skilled parent may choose any $k \geq \tilde{k}$ units of labor time for the education of his child. By contrast, the amount of k of an unskilled parent is only chosen at $\tilde{k}$. As noted by Razin and Sadka (1995), the Becker–Lewis model does not necessarily imply that richer people have fewer children. We therefore need the above additional assumption to yield more empirically plausible implications. Moreover, it should be noted that our results will hold qualitatively as long as it is assumed that unskilled parents' marginal contribution of their time to child education is substantially lower than that of skilled parents. This assumption and the implication receive supporting empirical evidence in Lynch (2000) and Ramey and Ramey (2010): college-educated mothers spend much more time educating their children than less-educated mothers in the USA. In fact, essentially a very similar assumption is made in Moav (2005). Thus, it is merely to save algebra that we assume that unskilled parents' marginal contribution of their time to child education is zero beyond a certain level in Assumption 1. In contrast to the existing literature, the model setup in this section implies that under both public and private educational systems, more skilled parents have fewer children with higher quality.

Let a skilled parent's income be $w_{\mathrm{s}}(1 - \xi n - kn)$ and an unskilled parent's income be $w_{\mathrm{u}}(1 - \xi n - \tilde{k}n)$ where w_{s} and w_{u} are the wage rates of a skilled and unskilled worker, respectively. Throughout this paper, we assume that

[5]Making this assumption is a simple yet relevant way to capture the stylized fact that better educated parents spend more time in the education of each child according to the empirical evidence in the literature of both sociology and economics (e.g., Ballantine 2001, Weinberg 2001, and Ramey and Ramey 2010).

the wage rates are fixed and that $w_s > w_u$.[6] The wage income is spent on own consumption c and education d per child in the private education regime:

$$c = (1 - \xi n - kn)\, w_s - dn \tag{2}$$

for a skilled worker and

$$c = \left(1 - \xi n - \tilde{k}n\right) w_u - dn \tag{3}$$

for an unskilled worker who always chooses $k = \tilde{k}$ under Assumption 1. We will also consider public education financed by income taxes later.

In line with Assumption 1, the education of a child is further specified to be a Cobb–Douglas function of a parental time input and a physical input according to

$$e_s = \gamma_s k_s^{\delta} d_s^{1-\delta} \text{ for } \quad k_s \geq \tilde{k}; \quad e_u = \gamma_u \tilde{k}_u^{\delta}\, d_u^{1-\delta} \text{ for } k_u \geq \tilde{k}, \quad \gamma_s \geq \gamma_u, \tag{4}$$

where $\delta \in (0, 1)$ is a share parameter measuring the relative importance of time and physical inputs in education and subscripts s and u refer to skilled and unskilled parents, respectively. The justification for $\gamma_s \geq \gamma_u$ is as follows: Given the amount of time and income spent on educating a child, the children of skilled individuals, being exposed to a different value system, cultural values, and academic orientation, are more likely to be skilled themselves.[7] The lower bound $\tilde{k}$ is regarded as exogenous, standing for some minimum parental time supporting child education such as sending children to school and monitoring children for their home studies. Specifically, we assume that

Assumption 2

$$0 < \tilde{k} < \alpha\delta\xi \big/ (1 - \alpha\delta).$$

This assumption ensures that skilled parents are willing to choose $k > \tilde{k}$. Our assumption of two inputs in child education differs from Moav (2005) where only effective parental labor is used instead, and from de la Croix and Doepke (2003, 2004) where the same education spending per child determined by average human capital applies to all families.

[6] The wage rates can be conveniently endogenized in a small open economy framework (e.g., Galor and Zeira 1993 and Hazan and Berdugo 2002).

[7] The importance of family background, and particularly of parental academic achievements and human capital, for an individual's educational attainment has been consistently confirmed in the empirical literature; see Becker (1991) and de la Croix and Michel (2002) for useful surveys, and a more recent empirical study by Anger and Heineck (2010).

3 Population dynamics

To ease the exposition that follows, we introduce the following notations:

- λ_t The proportion of skilled individuals in the population at time t;
- p The probability of a child whose parent is skilled to become skilled;
- q The probability of a child whose parent is unskilled to become skilled;
- n_{s} The fertility of a skilled individual;
- n_{u} The fertility of an unskilled individual; and
- N_t The total size of cohort at time t.

As will be clear, four of the above six variables are time invariant. Thus, there is no time script for them to save notations. The probabilities of becoming skilled are functions of education outcome, $p = p(e_{\mathrm{s}}) \in (0, 1)$ and $q(e_{\mathrm{u}}) \in (0, 1)$, which are strictly increasing and strictly concave. These probabilities of children becoming skilled as a function of education inputs allow our model to address intergenerational mobility together with differential investment in education and differential fertility. By contrast, there is no intergenerational mobility in de la Croix and Doepke (2003, 2004) and Moav (2005) in which differential fertility is studied along with differential education investment.

We will derive the following relationships later:

$$p > q, \quad n_{\mathrm{s}} < n_{\mathrm{u}}.$$

At time $t+1$, the number of skilled workers whose parents are skilled is

$$p n_{\mathrm{s}} \lambda_t N_t, \tag{5}$$

while the number of skilled workers whose parents are unskilled is

$$q n_{\mathrm{u}} (1 - \lambda_t) N_t. \tag{6}$$

Therefore, the total number of skilled workers at time $t+1$ is

$$p n_{\mathrm{s}} \lambda_t N_t + q n_{\mathrm{u}} (1 - \lambda_t) N_t. \tag{7}$$

The working population at time $t+1$ is

$$n_{\mathrm{s}} \lambda_t N_t + n_{\mathrm{u}} (1 - \lambda_t) N_t.$$

Thus, the ratio of the skilled to working population at time $t+1$ is

$$\lambda_{t+1} = \frac{p n_{\mathrm{s}} \lambda_t N_t + q n_{\mathrm{u}} (1 - \lambda_t) N_t}{n_{\mathrm{s}} \lambda_t N_t + n_{\mathrm{u}} (1 - \lambda_t) N_t} = \frac{p n_{\mathrm{s}} \lambda_t + q n_{\mathrm{u}} (1 - \lambda_t)}{n_{\mathrm{s}} \lambda_t + n_{\mathrm{u}} (1 - \lambda_t)}. \tag{8}$$

This transition equation implies the following:

Lemma 1 *Suppose that $p > q$ and that $n_{\mathrm{u}} > n_{\mathrm{s}}$. If p, q, n_{s} and n_{u} are independent of λ_t, then the ratio of skilled to working population converges globally toward a unique steady state $q < \lambda^* < p$.*

Proof Differentiating Eq. 8 with respect to λ_t yields

$$\frac{\partial\lambda_{t+1}}{\partial\lambda_t} = \frac{n_{\mathrm{u}}n_{\mathrm{s}}(p-q)}{[n_{\mathrm{s}}\lambda_t + n_{\mathrm{u}}(1-\lambda_t)]^2} > 0,$$

$$\frac{\partial^2\lambda_{t+1}}{\partial\lambda_t^2} = -\frac{2n_{\mathrm{u}}n_{\mathrm{s}}(p-q)(n_{\mathrm{s}}-n_{\mathrm{u}})}{[n_{\mathrm{s}}\lambda_t + n_{\mathrm{u}}(1-\lambda_t)]^3} > 0.$$

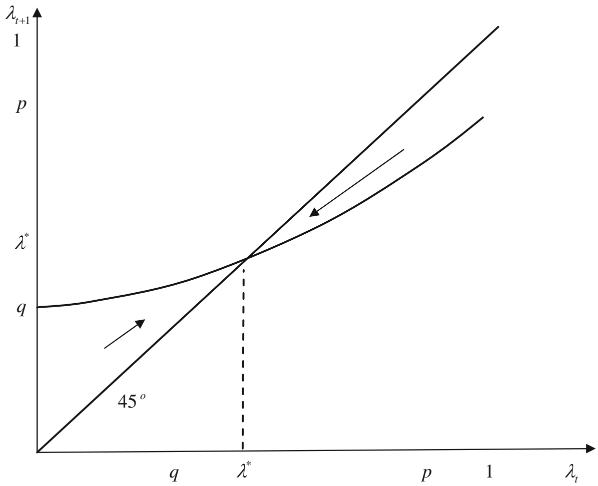
At $\lambda_t = 0$, $\lambda_{t+1} = q$. At $\lambda_t = 1$, $\lambda_{t+1} = p$. Consequently, there is a unique steady state $q < \lambda^* < p$. This steady state is globally stable because the slope $\partial\lambda_{t+1}/\partial\lambda_t$ must be smaller than 1 at the steady state: the curve of the transition equation starts above the 45° line at $\lambda_t < \lambda^*$ and finishes below the 45° line at $\lambda_t > \lambda^*$ as shown in Fig. 1. □

Now, we use y_t^p to denote an individual's income with one unit of labor fully supplied to the labor market, which we call potential income. Then, we define average or expected potential income permissible by labor time endowment, $E_t\left(y_t^p\right)$, and the variance of potential income, $\mathrm{Var}_t\left(y_t^p\right)$. The variance of actual income is to be defined and used as a measure of income inequality later, which depends on the variance of potential income. Given any λ_t in period t, the average potential income is equal to

$$\begin{aligned} E_t\left(y_t^p\right) &\equiv \lambda_t w_{\mathrm{s}} + (1-\lambda_t) w_{\mathrm{u}} \\ &= \lambda_t (w_{\mathrm{s}} - w_{\mathrm{u}}) + w_{\mathrm{u}}. \end{aligned} \tag{9}$$

Thus, the greater is the ratio of the skilled to working population λ_t, the higher is the average potential income.

Fig. 1 Transition of the ratio of skilled to working population



The variance of potential income is equal to

$$\begin{aligned}\mathrm{Var}_t\left(y_t^p\right) &= \lambda_t\left[w_\mathrm{s} - E_t\left(y_t^p\right)\right]^2 + (1-\lambda_t)\left[w_\mathrm{u} - E_t\left(y_t^p\right)\right]^2 \\ &= \lambda_t\left(1-\lambda_t\right)\left(w_\mathrm{s} - w_\mathrm{u}\right)^2. \end{aligned} \tag{10}$$

It implies that the level of potential-income inequality is increasing in the wage differential and dependent on the ratio of skilled to total population. The ratio of the skilled to working population may be regarded as the fraction of workers with secondary education or higher. If so, it is below 50 % in many low-income countries but it is typically above 50 % in high-income countries. When $\lambda_t < 1/2$, potential-income inequality is increasing in λ_t; when $\lambda_t = 1/2$, potential-income inequality peaks; and when $\lambda_t > 1/2$, potential-income inequality is decreasing in λ_t.

Also, as in Iyigun (1999), we use the odds ratio—the relative odds of being skilled for the children of unskilled parents compared with the children of skilled parents—as the measure of intergenerational mobility. Let M denote this ratio:

$$M \equiv \frac{\text{Probability}\left[\text{child is skilled} \mid \text{parent is unskilled}\right]}{\text{Probability}\left[\text{child is skilled} \mid \text{parent is skilled}\right]} = \frac{q}{p}.$$

The (relative) numbers of parents whose children experience mobility are upward mobility $q(1\text{-}\lambda_t)$ and downward mobility $(1\text{-}p)\lambda_t$.

We assume that the probabilities p and q are

$$p = 1 - \exp\left(-e_\mathrm{s}\right), \tag{11}$$

$$q = 1 - \exp\left(-e_\mathrm{u}\right), \tag{12}$$

where e_s is the education level of children with skilled parents and e_u is the education level of children with unskilled parents in Eq. 4. It should be noted that our formulation incorporates the idea that a child's educational outcome is determined by some stochastic factors as well as parental and societal inputs, which is in line with the recent literature.[8]

4 Private education

In the private education regime, the problem of a skilled worker maximizing utility in Eq. 1 subject to Eqs. 2 and 4 can be expressed as an unconstrained problem:

$$\max\left\{\ln(n) + \alpha\delta\ln(k) + \alpha(1-\delta)\ln(d) + \beta\ln\left[(1-\xi n - kn)\,w_\mathrm{s} - dn\right] + \alpha\ln\gamma\right\} \tag{13}$$

[8]See, e.g., Cremer et al. (2006), Mookherjee and Napel (2007), and Fan and Stark (2008).

by choice of (d, k, n), taking w_s and γ as given. From Eq. 13, the first-order conditions are

$$n: \quad \frac{1}{n} = \frac{\beta\left[(\xi + k)\, w_{\mathrm{s}} + d\right]}{(1 - \xi n - kn)\, w_{\mathrm{s}} - dn}, \tag{14}$$

$$k: \quad \frac{\alpha\delta}{k} = \frac{\beta n w_{\mathrm{s}}}{(1 - \xi n - kn)\, w_{\mathrm{s}} - dn}, \tag{15}$$

$$d: \quad \frac{\alpha\,(1-\delta)}{d} = \frac{\beta n}{(1 - \xi n - kn)\, w_{\mathrm{s}} - dn}. \tag{16}$$

The seemingly simple problem may have no solution because the budget constraint has non-convex components dn and kn. To ensure an interior optimal solution, we make the following assumption throughout the paper:

Assumption 3

$$0 < \alpha < 1.$$

From the utility function in Eq. 1, this assumption corresponds to the case where the taste for the education of children is weaker than the taste for the number of children in line with similar assumptions in Ehrlich and Lui (1991) and Zhang et al. (2001). Similarly, one can derive the first-order conditions with respect to n and d for an unskilled worker:

$$n: \quad \frac{1}{n} = \frac{\beta\left[\left(\xi + \tilde{k}\right) w_{\mathrm{u}} + d\right]}{\left[1 - \left(\xi + \tilde{k}\right) n\right] w_{\mathrm{u}} - dn}, \tag{17}$$

$$d: \quad \frac{\alpha\,(1-\delta)}{d} = \frac{\beta n}{\left[1 - \left(\xi + \tilde{k}\right) n\right] w_{\mathrm{u}} - dn}. \tag{18}$$

As mentioned earlier, under Assumption 1, an unskilled worker always chooses $k = \tilde{k}$.

Using subscripts s and u for skilled and unskilled workers, respectively, we give the solutions below and relegate the proof to Electronic supplementary material—Appendix A:

Lemma 2 *Under* $0 < \alpha < 1$*, in the private education regime, there is a unique optimal interior solution to the problem of a skilled worker*

$$n_{\mathrm{s}} \equiv \frac{(1-\alpha)}{\xi\,(1+\beta)}; \quad k_{\mathrm{s}} \equiv \frac{\alpha\delta\xi}{(1-\alpha)}; \quad d_{\mathrm{s}} \equiv \frac{\alpha\xi\,(1-\delta)}{1-\alpha} w_{\mathrm{s}}$$

and an unskilled worker

$$n_{\mathrm{u}} \equiv \frac{1-\alpha\,(1-\delta)}{\left(\xi+\tilde{k}\right)(1+\beta)}; \quad k_{\mathrm{u}} = \tilde{k}; \quad d_{\mathrm{u}} \equiv \frac{\alpha\,(1-\delta)\left(\xi+\tilde{k}\right)}{1-\alpha\,(1-\delta)} w_{\mathrm{u}}.$$

Note that Assumption 2, $\tilde{k} < \alpha\delta\xi \big/ (1-\alpha\delta)$, implies $\tilde{k} < k_{\mathrm{s}} = \alpha\delta\xi \big/ (1-\alpha)\tilde{k}$. In addition, comparing the fertility rates and the fractions of time and income spent on the education of each child of skilled and unskilled workers leads to

Proposition 1 *In the private education regime, a skilled worker has fewer children but invests a greater fraction of income and time in the education of each child than an unskilled worker:*

$$n_{\mathrm{s}} < n_{\mathrm{u}}; \quad \frac{d_{\mathrm{s}}}{w_{\mathrm{s}}} > \frac{d_{\mathrm{u}}}{w_{\mathrm{u}}}; \quad k_{\mathrm{s}} > k_{\mathrm{u}} = \tilde{k}.$$

Proof The difference $n_{\mathrm{u}} - n_{\mathrm{s}}$ is signed by

$$n_{\mathrm{u}} - n_{\mathrm{s}} = \frac{1-\alpha\,(1-\delta)}{\left(\xi+\tilde{k}\right)(1+\beta)} - \frac{1-\alpha}{\xi\,(1+\beta)}$$

$$= \frac{1}{(1+\beta)\,\xi\left(\xi+\tilde{k}\right)} \left\{\xi\,[1-\alpha\,(1-\delta)] - (1-\alpha)\left(\xi+\tilde{k}\right)\right\}$$

$$= \frac{1}{(1+\beta)\,\xi\left(\xi+\tilde{k}\right)} \left[\alpha\delta\xi - (1-\alpha)\,\tilde{k}\right] > 0$$

under Assumption 2, i.e., $\tilde{k} < \alpha\delta\xi/(1-\alpha\delta)$. The difference $d_{\mathrm{s}}/w_{\mathrm{s}} - d_{\mathrm{u}}/w_{\mathrm{u}}$ is given by

$$\frac{d_s}{w_s} - \frac{d_u}{w_u} = \frac{\alpha\,(1-\delta)\,\xi}{1-\alpha} - \frac{\alpha\,(1-\delta)\left(\xi+\tilde{k}\right)}{1-\alpha\,(1+\delta)}$$

$$= \frac{\alpha\,(1-\delta)}{(1-\alpha)\,[1-\alpha\,(1-\delta)]} \left\{\xi\,[1-\alpha\,(1-\delta)] - (1-\alpha)\left(\xi+\tilde{k}\right)\right\}$$

$$= \frac{\alpha\,(1-\delta)}{(1-\alpha)\,[1-\alpha\,(1-\delta)]} \left[\alpha\delta\xi - (1-\alpha)\,\tilde{k}\right] > 0$$

under the same condition $\tilde{k} < \alpha\delta\xi \big/ (1-\alpha\delta)$. Finally, Assumption 2 and Lemma 2 imply $k_{\mathrm{s}} > k_{\mathrm{u}} = \tilde{k}$. □

The reason why a skilled worker has lower fertility but spends greater fractions of time and income on the education of each child than an unskilled worker is mainly because the marginal contribution of parental time to child education is greater from the former than from the latter. This result differs from that in de la Croix and Doepke (2003) where differential fertility hinges on their assumption that the cost of education is fixed and does not depend on the parent's wage. Poor parents in their model have relatively higher costs for child education and therefore spend less income on child education and have more children. Our result also differs from that in Moav (2005) where differential fertility arises from the assumption that parents with sufficiently low human capital make no investment of time in child education.

We use the superscript "r" to denote the case of private education. Then, we have

$$n_{\mathrm{s}}^{\mathrm{r}} = \frac{(1-\alpha)}{\xi\,(1+\beta)}, \tag{19}$$

$$n_{\mathrm{u}}^{\mathrm{r}} = \frac{1-\alpha\,(1-\delta)}{\left(\xi+\tilde{k}\right)(1+\beta)}, \tag{20}$$

$$p^{\mathrm{r}} = 1-\exp\left\{-\gamma_{\mathrm{s}}\left(\frac{\alpha\delta\xi}{1-\alpha}\right)^{\delta}\left[\frac{\alpha\xi\,(1-\delta)}{1-\alpha}w_{\mathrm{s}}\right]^{1-\delta}\right\}, \tag{21}$$

$$q^{\mathrm{r}} = 1-\exp\left\{-\gamma_{\mathrm{u}}\tilde{k}^{\delta}\left[\frac{\alpha\,(1-\delta)\left(\xi+\tilde{k}\right)}{1-\alpha\,(1-\delta)}w_{\mathrm{u}}\right]^{1-\delta}\right\}. \tag{22}$$

According to Proposition 1, we have $n_{\mathrm{s}}^{\mathrm{r}} < n_{\mathrm{u}}^{\mathrm{r}}$ and $k_{\mathrm{s}} > \tilde{k}$. From $w_s > w_u$ and from Proposition 1, it follows immediately that the amount of education spending per child with a skilled parent must be greater than that with an unskilled parent. Thus, $e_{\mathrm{s}} > e_{\mathrm{u}}$ and $p^{\mathrm{r}} > q^{\mathrm{r}}$. In a nutshell, in the private education regime, a skilled parent has fewer children and gives each of them a greater chance to become skilled than an unskilled parent.

Clearly, Lemma 1 applies to the case with private education in our paper. That is, the ratio of the skilled to working population is convergent globally to a unique steady state. Plausible parameterizations can construct $q < 1/2 < p$ and fertility rates such that the ratio of the skilled to working population converges toward a level above 1/2 from a starting level below 1/2. In the proof of Lemma 1, the differential probabilities for children to become skilled in favor of those with skilled parents, $p > q$, are essential for the future ratio of the skilled to working population to change and converge to the steady state. Together with $p > q$, the differential fertility inversely related to the probability to become skilled, $n_{\mathrm{u}} > n_{\mathrm{s}}$, is useful to yield a relatively slower change in the ratio of the skilled to working population if the economy starts from a lower ratio of skilled to working population. This helps to explain the slow convergence found in the literature on growth empirics across nations (see, e.g., Barro and Sala-i-Martin 1992).

We now define actual income as $y_s = (1 - \xi n_s - k_s n_s) w_s$ and $y_u = (1 - \xi n_u - \tilde{k} n_u) w_u$ for a skilled and unskilled worker, respectively. From Lemma 2, $1 - \xi n_s - k_s n_s = 1 - \xi n_u - \tilde{k} n_u = [\beta + \alpha(1-\delta)]/(1+\beta)$. Thus, given any λ_t in period t, the average (actual) income is

$$E_t(y_t) \equiv \lambda_t (1 - \xi n_s - k_s n_s) w_s + (1 - \lambda_t)\left(1 - \xi n_u - \tilde{k} n_u\right) w_u = \left[\frac{\beta + \alpha(1-\delta)}{1+\beta}\right] E_t\left(y_t^p\right) \tag{23}$$

where $E(y^p) = \lambda w_s + (1-\lambda) w_u$ refers to the average potential income given in Eq. 9. Thus, the greater is the ratio of skilled to working population λ, the higher is the average income, as one may expect.

The variance of (actual) income, a measure of income inequality here, equals

$$\mathrm{Var}_t(y_t) = \lambda_t \left[w_s - E_t(y_t)\right]^2 + (1-\lambda_t)\left[w_u - E_t(y_t)\right]^2 = \left[\frac{\beta + \alpha(1-\delta)}{1+\beta}\right]^2 \lambda_t (1-\lambda_t)(w_s - w_u)^2 . \tag{24}$$

Now, we assume that the starting point of the economy is when λ is less than 0.5. This appears to be a reasonable assumption, since the skilled to unskilled ratio of an economy is usually low at its early stage of development. Then, Eq. 24 implies that the level of income inequality is increasing in the wage differential and dependent on the ratio of skilled to working population. When $\lambda_t < 1/2$, income inequality is increasing in λ_t; when $\lambda_t = 1/2$, income inequality peaks; and when $\lambda_t > 1/2$, income inequality is decreasing in λ_t toward its steady-state value. On the other hand, average fertility $\lambda_t n_s + (1-\lambda_t) n_u$ is falling with λ_t since $n_s < n_u$ in Proposition 1. The unique steady-state level of λ_t will be established in Section 7.

We summarize the results below:

Proposition 2 *As the ratio of skilled to working population λ_t rises from below 1/2 and converges to a level above 1/2, average income rises, average fertility falls, and income inequality first rises for $\lambda_t < 1/2$, peaks at $\lambda_t = 1/2$, and falls to its long-run steady-state value for $\lambda_t > 1/2$.*

Proposition 2 complements the existing literature by presenting the pattern of change in income inequality in the development process that resembles the Kuznets hypothesis from a different angle.[9] The process is driven by

[9] For some related existing models, see Galor and Tsiddon (1996) and the literature surveyed wherein. In those articles, yet, the economic forces underlying the result resembling the Kuznets curve are different from the one in action here. Specifically, in the previous literature, the curve emerges because of technological progress, which is not the case here.

the rising convergent ratio of skilled to working population in our model with fixed wage rates. Since the poor has more offspring in our model, the distribution of income will change over time regardless of mobility between groups. Also, this result is in contrast to many models on education, inequality, and intergenerational mobility in which inequality is an exogenous monotonic process over time (e.g., Glomm and Ravikumar 1992; Benabou 1996; Zhang 1996; Davies et al. 2005). In our model with exogenously fixed wage rates, λ_t varies over time (and hence $\mathrm{Var}_t(y_t)$), depending on the fertility rates and education investment of skilled and unskilled workers; and inequality follows an inverted-U-shaped path. Moreover, as λ_t rises over time, average income will increase and average fertility will decrease. This rising trend in average income and this falling trend in average fertility are also consistent with their observed trends in the last two centuries. Our results concerning the Kuznets curve are based on the assumption of fixed wage rates, which should be acknowledged as a caveat of this proposition. In future research, it will be interesting to explore the implications of more complicated dynamic structures of the wage rates.

5 Public education

In the case of public education, as in de la Croix and Doepke (2004), we assume that educational expenditure is chosen by the government and is financed by flat-rate taxation. Let τ denote the tax rate. Then, the budget constraint of a skilled worker is

$$c = (1 - \xi n - kn)\, w_{\mathrm{s}}\, (1 - \tau)\,, \tag{25}$$

whereas the budget constraint of an unskilled worker is

$$c = \left(1 - \xi n - \tilde{k}\, n\right) w_{\mathrm{u}}\, (1 - \tau)\,. \tag{26}$$

The education function is now

$$e_{\mathrm{s}} = \gamma_{\mathrm{s}} k_{\mathrm{s}}^{\delta}\, g_t^{1-\delta} \text{ for } k_{\mathrm{s}} \geq \tilde{k}; \quad e_{\mathrm{u}} = \gamma_{\mathrm{u}} \tilde{k}_{\mathrm{u}}^{\delta} g_t^{1-\delta} \text{ for } k_{\mathrm{u}} \geq \tilde{k} \tag{27}$$

where g is the government education spending per child. Again, any level of k such that $k \geq \tilde{k}$ may be chosen by a skilled worker and $k = \tilde{k}$ is always chosen by an unskilled worker. Every worker takes g as given. Also, in this section, we treat g as a government policy parameter. However, we will endogenize g in the next section.

The problem of a skilled worker is to maximize Eq. 1 subject to Eqs. 25 and 27, choosing k and n. The first-order conditions are

$$k: \quad \frac{\alpha\delta}{k} = \frac{\beta n}{1 - \xi n - kn}, \tag{28}$$

$$n: \quad \frac{1}{n} = \frac{\beta\,(\xi + k)}{1 - \xi n - kn}. \tag{29}$$

Similarly, the problem of an unskilled worker is to maximize Eq. 1 subject to Eqs. 26 and 27, choosing $k = \overline{k}$ and n. The first-order condition is

$$n: \quad \frac{1}{n} = \frac{\beta\left(\xi + \tilde{k}\right)}{1 - \left(\xi + \tilde{k}\right)n}. \tag{30}$$

From these first-order conditions, the solutions are given below:

Lemma 3 *In the public education regime, there is a unique optimal interior solution*

$$n_{\rm s} \equiv \frac{(1-\alpha\delta)}{\xi\,(1+\beta)}; \quad k_{\rm s} \equiv \frac{\alpha\delta\xi}{1-\alpha\delta}$$

for a skilled worker and

$$n_{\rm u} = \frac{1}{\left(\xi + \tilde{k}\right)(1+\beta)}; \quad k_{\rm u} = \tilde{k}$$

for an unskilled worker.

Proof The solutions are from the first-order conditions above for a skilled or an unskilled worker. Since $0 < \delta < 1$ in Eq. 4 and $0 < \alpha < 1$ in Assumption 3, it follows that $0 < \alpha\delta < 1$. Thus, the solution here is interior. The derivation of the sufficient conditions for the solutions to be optimal is similar to that in the proof of Lemma 2. □

Here, $k_{\rm s} > \tilde{k}$ for a skilled worker under Assumption 2, $\tilde{k} < \alpha\delta\xi\big/(1-\alpha\delta)$, in the public education regime.

From Lemma 3, we have the following proposition:

Proposition 3 *In the public education regime, a skilled worker has fewer children and spends more time on the education of each child than an unskilled worker:*

$$n_s < n_u; \quad k_s > \tilde{k}.$$

Proof From Lemma 3, we observe that

$$\begin{aligned} n_{\rm u} - n_{\rm s} &= \frac{1}{\left(\xi + \tilde{k}\right)(1+\beta)} - \frac{1-\alpha\delta}{\xi\,(1+\beta)} \\ &= \frac{1}{(1+\beta)\,\xi\left(\xi + \tilde{k}\right)}\left[\xi - (1-\alpha\delta)\left(\xi + \tilde{k}\right)\right] \\ &= \frac{1}{(1+\beta)\,\xi\left(\xi + \tilde{k}\right)}\left[\alpha\delta\xi - \tilde{k}\,(1-\alpha\delta)\right] > 0 \end{aligned}$$

under Assumption 2, $\tilde{k} < \alpha\delta\xi/(1-\alpha\delta)$. Assumption 2 and Lemma 3 also imply $k_s > \tilde{k}$. □

The reason for a skilled worker to have lower fertility and to spend more time on the education of each child than an unskilled worker is the same as in the case with private education. The only difference is that the cost of public education spending is the proportional tax which remains different between skilled and unskilled parents, unlike the fixed and equal cost of education spending in de la Croix and Doepke (2003).

Lemma 1 applies to the public education regime in a special case with fixed government spending on education. In this case, the tax rate for government education spending has to change along with the ratio of the skilled to working population in a general equilibrium. However, such a change in the tax rate has no effect on fertility and the time input for education in our model as shown in Lemma 2. Consequently, the equilibrium solution for fertility rates, time inputs, and the probabilities for children to become skilled are independent of the ratio of skilled to working population in this special case.

We now compare across education regimes with superscripts r and g for private and public regimes, respectively. From Lemmas 1 and 2, we observe the following:

Proposition 4 *For both skilled and unskilled workers, fertility is higher in the public than in the private education regime; for skilled workers, their time input for the education of each child is lower in the public than in the private education regime:*

$$n_s^r < n_s^g; \quad n_u^r < n_u^g;$$
$$k_s^r > k_s^g.$$

The main intuition for the results in Proposition 4 is as follows: First, in the public education regime, the amount of education spending g is outside the control of individuals. Thus, skilled workers cannot make complementary increases in their time and income for the education of their children to the levels they would otherwise like to achieve in the private education regime. This tends to reduce the amount of time skilled workers use for their children's education. Second, the labor income tax reduces the after-tax opportunity cost of spending time rearing children (i.e., the substitution effect), thereby tending to raise fertility for both types of workers compared to the levels of fertility in the private education regime. Also, this tax substitution effect tends to reduce the time for child education by skilled workers in the public education regime.[10] The negative tax effect on the time input in education is essentially

[10] A further comment is that this result may be related to our simplifying assumption that public education cannot be complemented by private investment. The results may be modified if this assumption is relaxed, and de la Croix and Doepke (2009) provide a framework for such a possible extension.

similar to that captured in Glomm and Ravikumar (1992) with fixed fertility. With endogenous fertility, the positive effect of public education on fertility also emerges in de la Croix and Doepke (2004) under the same assumption that parents are not allowed to provide private education in addition to the common level of public education spending. We will allow skilled parents to opt out of public education after paying the tax for public education later.

We use the superscript "g" to denote the case of public education. Then, we have

$$n_s^g = \frac{(1-\alpha\delta)}{\xi\,(1+\beta)}, \tag{31}$$

$$n_u^g = \frac{1}{\left(\xi + \tilde{k}\right)(1+\beta)}, \tag{32}$$

$$p^g = 1 - \exp\left[-\left(\frac{\alpha\delta\xi}{1-\alpha\delta}\right)^{\delta} g^{1-\delta}\gamma_s\right], \tag{33}$$

$$q^g = 1 - \exp\left(-\tilde{k}^{\delta} g^{1-\delta}\gamma_u\right). \tag{34}$$

Considering the ratio of fertility of an unskilled worker to fertility of a skilled worker across the two regimes, we have the following result:

Proposition 5 *The ratio of fertility of an unskilled worker to fertility of a skilled worker is smaller in the public education regime than in the private education regime:*

$$\frac{n_u^g}{n_s^g} < \frac{n_u^r}{n_s^r}.$$

Proof From Lemma 2, $n_u^r/n_s^r = \xi[1-\alpha(1-\delta)]/[(1-\alpha)(\xi+\tilde{k})]$. Similarly, from Lemma 3, $n_u^g/n_s^g = \xi/[(1-\alpha\delta)(\xi+\tilde{k})]$. Thus, we have $(n_u^g/n_s^g) - (n_u^r/n_s^r) = -\xi\alpha^2\delta(1-\delta)/[(\xi+\tilde{k})(1-\alpha)(1-\alpha\delta)] < 0$ since $\alpha < 1$ under Assumption 3 and $0 < \delta < 1$ in Eq. 4. □

Proposition 5 means that public education reduces the relative gap between fertility rates of unskilled and skilled workers. This is similar to that in de la Croix and Doepke (2004) but arises from a different channel. The main reason for this result in our model is that public education reduces the time for child education by skilled parents in Proposition 4, which is the chief factor for the existence of fertility differentials between skilled and unskilled workers in our model. In de la Croix and Doepke (2004), however, there is no time input in education.

Moreover, we would like to comment that the growth implication of public education may be ambiguous since public education may reduce skilled individuals' time of working. The study of growth implications of public education entails the analysis of optimal expenditure of public education, which is discussed in the following sections.

Given any λ_t in period t, the average (actual) income is

$$E_t\left(y_t^{\mathrm{g}}\right) \equiv \lambda_t\left(1-\xi n_{\mathrm{s}}^{\mathrm{g}}-k_{\mathrm{s}}^{\mathrm{g}} n_{\mathrm{s}}^{\mathrm{g}}\right) w_{\mathrm{s}}+\left(1-\lambda_t\right)\left(1-\xi n_{\mathrm{u}}^{\mathrm{g}}-\tilde{k} n_{\mathrm{u}}^{\mathrm{g}}\right) w_{\mathrm{u}}$$
$$=\frac{\beta}{1+\beta} E_t\left(y_t^p\right), \tag{35}$$

where $E\left(y^p\right)=\lambda w_{\mathrm{s}}+(1-\lambda) w_{\mathrm{u}}$ refers to the average potential income. Again, the greater is λ, the higher is the average income as in the private education regime.

The variance of (actual) income in the public education regime equals

$$\operatorname{Var}_t\left(y_t^{\mathrm{g}}\right)=\lambda_t\left[w_{\mathrm{s}}-E_t\left(y_t^{\mathrm{g}}\right)\right]^2+\left(1-\lambda_t\right)\left[w_{\mathrm{u}}-E_t\left(y_t^{\mathrm{g}}\right)\right]^2$$
$$=\left(\frac{\beta}{1+\beta}\right)^2 \lambda_t\left(1-\lambda_t\right)\left(w_{\mathrm{s}}-w_{\mathrm{u}}\right)^2 . \tag{36}$$

Thus, Proposition 2 also applies to the public education regime.

Comparing average income and inequality with the same λ_t in Eqs. 23, 24, 35, and 36, we observe the following:

Proposition 6 *Starting from the same initial ratio of skilled to working population, average income and income inequality are higher in the private education regime than in the public education regime in the short run.*

Proposition 6 implies that at any given time switching from private to public education will immediately reduce average income and income inequality. In fact, this result is also obtained in models of income inequality with fixed fertility (e.g., Glomm and Ravikumar 1992; Zhang 1996; Davies et al. 2005). A similar result also arises in a model with endogenous and differential fertility in de la Croix and Doepke (2004). However, the long-run levels of average income may be higher in the public than private education regime, which will be discussed in Section 7 where we examine the issues in the steady state.

Unskilled agents benefit from public education because of the redistributive mechanism as claimed in the related literature (e.g., de la Croix and Doepke 2003, 2004 or Cardak 2005, and the literature surveyed therein). However, there is nevertheless an interesting angle here as it helps the poor by allowing them to be more educated and thus experience upward mobility as well as increasing their fertility. On the other hand, skilled agents may pull their children out of public schools despite paying the tax. We will consider this scenario in the next section.

6 Intergenerational mobility under different educational systems

In this section, we first examine intergenerational mobility under different educational systems, which are first characterized by the following two propositions:

Proposition 7 *Under private education, intergenerational mobility q/p increases with w_u and $\tilde{k}$ and decreases with w_s.*

Proof From Lemma 2, we know that q increases with w_{u} and $\tilde{k}$ and is independent of w_{s}; p increases with w_{s} and is independent of w_{u} and $\tilde{k}$. Then, recall that intergenerational mobility is defined as the ratio between q and p, we have proved the proposition. □

The intuition of the above proposition is straightforward: First, in a system of private education, an increase in unskilled wage increases educational resources for the children with unskilled parents, which increases intergenerational mobility. Second, an increase in the skilled wage increases educational resources for the children with skilled parents, which decreases intergenerational mobility. Third, an increase in $\tilde{k}$ induces unskilled parents to spend more time educating their children and reduce fertility, which increases intergenerational mobility. Propositions 1 and 7 imply that when fertility differential between unskilled and skilled decreases due to a rise in $\tilde{k}$, intergenerational mobility increases. Given (q, p), when λ_t rises toward its steady state with private education, the relative number of unskilled parents whose children become skilled, $q(1-\lambda_t)$, declines, whereas the relative number of skilled parents whose children become unskilled, $(1-p)\lambda_t$, rises. The assumption of constant wage rates simplifies the comparative statics analysis greatly.

We now look at intergenerational mobility in the public education regime.

Proposition 8 *Under public education, intergenerational mobility q/p increases with $\tilde{k}$ and decreases with α, ξ, and δ.*

Proof From Lemma 3, we know that q increases with $\tilde{k}$ and is independent of α, ξ, and δ; p increases with α, ξ, and δ and is independent of $\tilde{k}$. Then, recall that intergenerational mobility is defined as the ratio between q and p, we have proved the proposition. □

The intuition of the above proposition is as follows: First, as described above, an increase in $\tilde{k}$ induces unskilled parents to spend more time educating their children and reduce fertility, which increases intergenerational mobility. Second, from Lemma 3, we can see that with an increase in α, ξ, or δ, the fertility of a skilled parent decreases and at the same time the skilled parent spends more time educating the children, which increases p and hence decreases intergenerational mobility. This proposition and Proposition 3 imply that a variation in $\tilde{k}$, α, or δ may have very different effects on the fertility differential and intergenerational mobility when public education spending is held constant. For fixed government education spending, q and p are independent of the ratio of skilled to working population. In this case, the relative number of families experiencing upward/downward intergenerational mobility varies with λ_t in the same way as in the private education regime. However, higher government education spending raises p and q, tending to raise upward mobility $q(1-\lambda_t)$ but reduce downward mobility $(1-p)\lambda_t$ in the short run. Moreover, higher government education spending may raise λ_{t+j} over time and may have a negative (positive) effect on the upward (downward) mobility (Fig. 2).

In order to compare intergenerational mobility across different education regimes, we now analyze a case of "optimal" public education. In period t, the government takes the initial λ_t as given and chooses the level of public education spending per child to maximize the average welfare of the working population:

$$W_t = \lambda_t U_{\mathrm{s},t} + (1-\lambda_t)\, U_{\mathrm{u},t} \tag{37}$$

where U_{s} and U_{u} stand for the utility of a skilled and an unskilled worker, respectively, based on the utility specification in Eq. 1. In so doing, the government knows the optimal solution to the problems of skilled and unskilled parents in Section 5 as a function of government education spending and the tax rate. The government runs a balanced budget in every period. From Lemma 3, $1-\left(\xi+k_{\mathrm{s}}^{\mathrm{g}}\right)n_{\mathrm{s}}^{\mathrm{g}} = 1-\left(\xi+\tilde{k}\right)n_{\mathrm{u}}^{\mathrm{g}} = \beta\big/(1+\beta)$. Then, the government budget constraint is given by

$$g_t\left[n_{\mathrm{s}}^{\mathrm{g}}\lambda_t + n_{\mathrm{u}}^{\mathrm{g}}(1-\lambda_t)\right] = \tau_t\left[\lambda_t w_{\mathrm{s}} + (1-\lambda_t)\, w_{\mathrm{u}}\right]\frac{\beta}{1+\beta}, \tag{38}$$

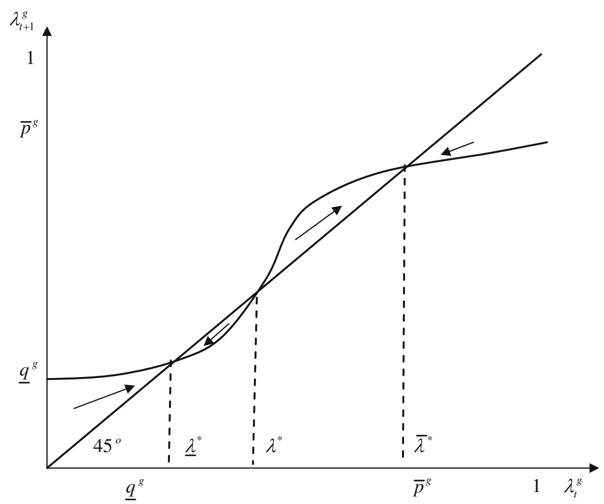


Fig. 2 Transition of the ratio of skilled to working population with optimal government education spending

where n_{s} and n_{u} are time invariant as given in Lemma 3. Using Eqs. 1, 27, 37, and the solutions for consumption, fertility, and education investment given earlier, the government's problem maximizing Eq. 37 subject to Eq. 38 is equivalent to

$$\max_{g_t} \ \alpha (1-\delta) \ln g_t + \beta \ln \left\{ \beta \left[\lambda_t w_{\mathrm{s}} + (1-\lambda_t) w_{\mathrm{u}} \right] - g_t (1+\beta) \left[n_{\mathrm{s}}^{\mathrm{g}} \lambda_t + n_{\mathrm{u}}^{\mathrm{g}} (1-\lambda_t) \right] \right\}$$

where the expression in the bracket {...} is the numerator of $1 - \tau_t$ according to Eq. 38. The essence of this problem is clearly a trade-off between the contribution of public spending to child education and the cost of doing so via the income tax.

The first-order condition of the government problem is given below:

$$\frac{\alpha (1-\delta)}{g_t} = \frac{\beta (1+\beta) \left[n_{\mathrm{s}}^{\mathrm{g}} \lambda_t + n_{\mathrm{u}}^{\mathrm{g}} (1-\lambda_t) \right]}{\beta \left[\lambda_t w_{\mathrm{s}} + (1-\lambda_t) w_{\mathrm{u}} \right] - g_t (1+\beta) \left[n_{\mathrm{s}}^{\mathrm{g}} \lambda_t + n_{\mathrm{u}}^{\mathrm{g}} (1-\lambda_t) \right]} \tag{39}$$

leading to the solution for g_t below:

Proposition 9 *The optimal education policy is given by*

$$\tau = \frac{\alpha (1-\delta)}{\beta + \alpha (1-\delta)} \in (0,1); \quad g_t = \frac{\alpha\beta (1-\delta) \left[\lambda_t w_{\mathrm{s}} + (1-\lambda_t) w_{\mathrm{u}} \right]}{(1+\beta) \left[\beta + \alpha (1-\delta) \right] \left[n_{\mathrm{s}}^{\mathrm{g}} \lambda_t + n_{\mathrm{u}}^{\mathrm{g}} (1-\lambda_t) \right]}. \tag{40}$$

The optimal tax rate is constant over time; the optimal education spending per child g_t is increasing in average potential income $\lambda_t w_s + (1-\lambda_t) w_u$, decreasing in average fertility $n_{\mathrm{s}}^{\mathrm{g}} \lambda_t + n_{\mathrm{u}}^{\mathrm{g}} (1-\lambda_t)$ and increasing in the ratio of skilled to working population.

Proof Here, sign $\partial g_t / \partial \lambda_t = (w_{\mathrm{s}} - w_{\mathrm{u}})[n_{\mathrm{s}}^{\mathrm{g}} \lambda_t + n_{\mathrm{u}}^{\mathrm{g}} (1-\lambda_t)] - (n_{\mathrm{s}}^{\mathrm{g}} - n_{\mathrm{u}}^{\mathrm{g}})[\lambda_t w_{\mathrm{s}} + (1-\lambda_t) w_{\mathrm{u}}] > 0$ since $w_{\mathrm{s}} > w_{\mathrm{u}}$ and $n_{\mathrm{s}}^{\mathrm{g}} < n_{\mathrm{u}}^{\mathrm{g}}$. The proof of other parts of the proposition is straightforward. □

The optimal income tax rate for public education is constant over time, increasing in the taste parameter for child education and the share parameter of education spending and decreasing in the taste parameter for consumption. The amount of optimal government spending on education is increasing in average potential income and decreasing in average fertility. One result is that the amount of optimal government education spending is increasing in the ratio of skilled to working population. This is consistent with the fact that developed countries having much higher ratios of skilled to working population also have greater public spending per child than developing countries. The dependence of public education spending on the ratio of skilled to working population via average income and via average fertility enriches the understanding of the underlying forces for government education spending compared to the literature.

In many countries education is mainly funded by the government (e.g., Glomm and Ravikumar 2003). Proposition 9 provides a rationale for this observed phenomenon. Also, note that this result is obtained even though public education diverts resources from the children with high-learning efficiency to the children with low-learning efficiency. Moreover, from the above proof, it is easy to see that we would get qualitatively similar results based on different formulations of the social welfare function that may include the welfare of children in the parental utility. In this sense, our simple specification of the social welfare function is mainly meant to keep the analysis tractable.

In Proposition 9, optimal government spending is increasing with the ratio of the skilled to working population. In Proposition 4, however, public education leads to higher fertility rates for all parents but lower time inputs of skilled parents for child education. As a result, the probabilities of children becoming skilled increase with the ratio of skilled to working population in the public education regime. But these probabilities may be lower in the public education regime than those in the private education regime. This positive dependence of such probabilities on the ratio of skilled to working population does not satisfy the conditions in Lemma 1 for global convergence to a unique steady state. Analytically, the dependence of the probabilities for children to become skilled on the ratio of skilled to working population complicates the issue of convergence in the ratio of skilled to working population. We will get back to this issue with optimally chosen government education spending later.

We now allow a skilled agent to deviate from the optimal public education policy, assuming that all other agents stay with it. A Nash equilibrium is reached if it satisfies certain conditions under which no skilled agent is willing to deviate. The deviation takes the form that anyone can opt out to seek private education for their children after paying the tax for public education. The justification is that schooling is full time for children, regardless of whether it is public or privatve, so that a child can only be placed in one type of school. The condition for a Nash equilibrium is given below.

Proposition 10 *No skilled agent is willing to opt out the optimal public education if*

$$\left(\frac{1-\alpha\delta}{1-\alpha}\right)^{\frac{1-\alpha}{\alpha(1-\delta)}}(1-\delta) > \left[\frac{\xi}{\left(\xi+\tilde{k}\right)(1-\alpha\delta)}\right]\frac{w_{\mathrm{s}}}{w_{\mathrm{u}}}.$$

Proof The budget constraint of a skilled agent opting out public education for their children is $c = (1 - \xi n - kn)\, w_{\mathrm{s}} (1 - \tau) - dn$ after paying the tax. The solution for n and k is the same as in the private education regime. The solution for consumption is the same regardless of whether the agent opts out public education:

$$c = \left(\frac{\beta}{1+\beta}\right) w_{\mathrm{s}} (1 - \tau) .$$

The solution for education spending is

$$d = \left[\frac{\alpha\xi (1-\delta)}{1-\alpha}\right] w_{\mathrm{s}} (1-\tau), \qquad d^{*} = \frac{\alpha\beta\xi w_{\mathrm{s}}}{(1-\alpha)\left[\beta + \alpha (1-\delta)\right]}.$$

Here, d^* is evaluated at the optimal tax rate τ^* given in Proposition 9.

For the same consumption with or without deviation, the welfare gain or loss of opting out public education is given by

$$U^{\mathrm{o}}_{\mathrm{s},t} - U^{\mathrm{g}}_{\mathrm{s},t} = \ln\left(n^{\mathrm{o}}_{\mathrm{s}}/n^{\mathrm{g}}_{\mathrm{s}}\right) + \alpha\delta \ln\left(k^{\mathrm{o}}_{\mathrm{s}}/k^{\mathrm{g}}_{\mathrm{s}}\right) + \alpha (1-\delta) \ln\left(d^{\mathrm{o}*}_{\mathrm{s}}/g*\right)$$

where superscripts o and g refer to opting out or staying with public education, respectively. Using Lemmas 1 and 2 and the solution for d^* above and for g^* in Proposition 9, the welfare gain or loss of opting out public education is rewritten as

$$\begin{aligned} U^{\mathrm{o}}_{\mathrm{s},t} - U^{\mathrm{g}}_{\mathrm{s},t} &= (1-\alpha\delta) \ln\frac{1-\alpha}{1-\alpha\delta} + \alpha (1-\delta) \\ &\quad \times \ln\left\{\left(\frac{1}{(1-\delta)(1-\alpha)}\right)\left[\frac{(1-\alpha\delta)\lambda_t + \frac{\xi}{\xi+\tilde{k}} (1-\lambda_t)}{\lambda_t + \frac{w_{\mathrm{u}}}{w_{\mathrm{s}}} (1-\lambda_t)}\right]\right\} \end{aligned}$$

which is negative if

$$\left(\frac{1-\alpha\delta}{1-\alpha}\right)^{\frac{1-\alpha}{\alpha(1-\delta)}} (1-\delta) > \frac{\lambda_t + \frac{\xi}{\left(\xi+\tilde{k}\right)(1-\alpha\delta)} (1-\lambda_t)}{\lambda_t + \frac{w_{\mathrm{u}}}{w_{\mathrm{s}}} (1-\lambda_t)} \equiv f(\lambda_t) .$$

The function $f(\lambda)$, as defined above, is decreasing:

$$f'(\lambda) = \left[\lambda + \frac{w_u}{w_s} (1-\lambda)\right]^{-2} \left[\frac{w_u}{w_s} - \frac{\xi}{\left(\xi + \tilde{k}\right)(1-\alpha\delta)}\right] < 0,$$

because $(\zeta + \tilde{k})(1-\alpha\delta) < (\xi + \frac{\alpha\delta\xi}{1-\alpha\delta})(1-\alpha\delta) = \xi$ under Assumption 2. So the maximum of $f(\lambda)$ is found at $\lambda = 0$, implying the condition in the Proposition. □

According to Proposition 10, skilled parents are less likely to opt out of public schools if the wage differential is smaller. In the USA, a vast majority of children have gone to public schools since the late 19th century as noted in Glomm and Ravikumar (1992). Therefore, the condition in Proposition 10 may be highly relevant.

Also, we have the following result on the comparison of intergenerational mobility across education regimes.

Proposition 11 *If the wage differential between skilled and unskilled workers is sufficiently large or if the taste for child education is sufficiently strong, then intergenerational mobility is greater in the public education regime than in the private education regime:*

$$M^{\mathrm{g}} > M^{\mathrm{r}}.$$

Proof It suffices to show the conditions for $M^{\mathrm{g}}/M^{\mathrm{r}} > 1$ whereby

$$\begin{aligned}\frac{M^{\mathrm{g}}}{M^{\mathrm{r}}} &= \frac{q^{\mathrm{g}}/p^{\mathrm{g}}}{q^{\mathrm{r}}/p^{\mathrm{r}}} \\ &= \frac{\left[1-\exp\left(-e_{\mathrm{u}}^{\mathrm{g}}\right)\right]/\left[1-\exp\left(-e_{\mathrm{s}}^{\mathrm{g}}\right)\right]}{\left[1-\exp\left(-e_{\mathrm{u}}^{\mathrm{r}}\right)\right]/\left[1-\exp\left(-e_{\mathrm{s}}^{\mathrm{r}}\right)\right]} \\ &= \frac{\left[1-\exp\left(-e_{\mathrm{u}}^{\mathrm{g}}\right)\right]\left[1-\exp\left(-e_{\mathrm{s}}^{\mathrm{r}}\right)\right]}{\left[1-\exp\left(-e_{\mathrm{u}}^{\mathrm{r}}\right)\right]\left[1-\exp\left(-e_{\mathrm{s}}^{\mathrm{g}}\right)\right]}.\end{aligned}$$

Clearly, $e_{\mathrm{u}}^{\mathrm{g}} > e_{\mathrm{u}}^{\mathrm{r}}$ and $e_{\mathrm{s}}^{\mathrm{r}} > e_{\mathrm{s}}^{\mathrm{g}}$ together are sufficient conditions for $M^{\mathrm{g}}/M^{\mathrm{r}} > 1$. Note that $e_{\mathrm{u}}^{\mathrm{g}} > e_{\mathrm{u}}^{\mathrm{r}}$ means $q_{\mathrm{u}}^{\mathrm{g}} > q_{\mathrm{u}}^{\mathrm{r}}$, that is, the upward mobility of children with unskilled parents is higher in the public than in the private education regime. Likewise, $e_{\mathrm{s}}^{\mathrm{r}} > e_{\mathrm{s}}^{\mathrm{g}}$ means $p_{\mathrm{s}}^{\mathrm{r}} > p_{\mathrm{s}}^{\mathrm{g}}$ or equivalently $1-p_{\mathrm{s}}^{\mathrm{g}} > 1-p_{\mathrm{s}}^{\mathrm{r}}$, that is, the downward mobility of children with skilled parents is higher in the public than in the private education regime. Since the time input chosen by unskilled workers for the education of each child is equal to the same exogenous amount $\tilde{k}$, $e_{\mathrm{u}}^{\mathrm{g}} > e_{\mathrm{u}}^{\mathrm{r}}$ is warranted by $g_t^{>} d_{\mathrm{u}}^{\mathrm{r}}$. Using the solutions for $d_{\mathrm{u}}^{\mathrm{r}}$ in Lemma 2 and for g_t in Eq. 40, the inequality $g_t^{>} d_{\mathrm{u}}^{\mathrm{r}}$ holds if

$$\frac{\lambda_t w_{\mathrm{s}} + (1-\lambda_t)\, w_{\mathrm{u}}}{w_{\mathrm{u}}} > \frac{[\beta + \alpha\,(1-\delta)]\left[(1-\alpha\delta)\left(\xi + \tilde{k}\right)\lambda_t + \xi\,(1-\lambda_t)\right]}{[1-\alpha\,(1-\delta)]\,\xi\beta}.$$

Similarly, using the solutions for $e_{\mathrm{s}}^{\mathrm{r}}$ from Eq. 4 and Lemma 2 and for $e_{\mathrm{s}}^{\mathrm{g}}$ from Eq. 27 and Lemma 3 and g_t in Eq. 40, the inequality $e_{\mathrm{s}}^{\mathrm{r}} > e_{\mathrm{s}}^{\mathrm{g}}$ holds if

$$\frac{w_{\mathrm{s}}}{\lambda_t w_{\mathrm{s}} + (1-\lambda_t)\, w_{\mathrm{u}}} > \frac{\beta\,(1-\alpha)^{1/(1-\delta)}\left(\xi + \tilde{k}\right)}{[\beta + \alpha\,(1-\delta)]\,(1-\alpha\delta)^{\delta/(1-\delta)}\left[(1-\alpha\delta)\left(\xi + \tilde{k}\right)\lambda_t + \xi\,(1-\lambda_t)\right]}.$$

Clearly, if the wage differential is sufficiently large, then $M^{\mathrm{g}} > M^{\mathrm{r}}$. Also, if α is large enough, or at least if 1-α is close enough to zero, then $M^{\mathrm{g}} > M^{\mathrm{r}}$. □

This result is obtained partly because the impact of parental income differential on children's education outcome is prevalent in the private education system but it is eliminated by public education, as in the literature. Intuitively, the equalization of education spending for every child by public education tends to raise the upward mobility of children with unskilled parents and the downward mobility of children with skilled parents. Also, public education raises the downward mobility of children with skilled parents by reducing the amount of time skilled parents spend on their children's education.

However, the new light this paper sheds on mobility comes from the negative response of the relative fertility differential between unskilled and skilled workers to public education (in Proposition 5). This reduced relative fertility differential in $n_{\mathrm{u}}/n_{\mathrm{s}}$ under public education means relatively more children from skilled workers compared to the private education regime and tends to raise the ratio of skilled to working population in future times, other things being equal. This rise in the skilled portion of the future working population can in turn raise average income and hence government spending on education in future times (particularly so if the wage differential is large). Subsequently, the rise in future government spending under public education will brighten the chance for children with unskilled parents to become skilled (upward mobility) and will also mitigate the increased downward mobility of children with skilled parents. As a result, switching from private to public education tends to raise the downward mobility of children with skilled parents more in the short run than in the long run on the one hand and raise the upward mobility of children with unskilled parents in all times on the other.

7 Steady state

We now provide a definition of the steady state of the population dynamics, which is similar to that of Mookherjee and Napel (2007), Fan and Stark (2008), and Cremer et al. (2011).

Definition 1 *The population dynamics of an economy is in a steady state if*

$$\lambda_t = \lambda_{t+1} = \lambda_{t+2} = \dots$$

It is easy to see that this definition implies that, in the steady state of the ratio of skilled to working population, the population growth rate is constant. To avoid the complexity of convergence in λ to the steady state in the public education regime, we focus on fixed government spending on education and allow the tax rate to change to run a balanced government budget in this section. As mentioned earlier, in our model, the tax rate change does not affect the equilibrium solution for fertility and the time input in child education. We now determine the steady state below.

Lemma 4 *The steady state in the private education regime and in the public education regime with fixed government spending on education is given by*

$$\lambda = \frac{n_u (1+q) - pn_s - \sqrt{\Delta}}{2(n_u - n_s)},$$

where $\Delta = \left[n_u (1+q) - pn_s\right]^2 - 4(n_u - n_s) n_u q$.

Proof In the private education regime and in the public education regime with fixed government spending on education, the equilibrium solutions for (n_s, n_u, p, q) are independent of λ. In these cases, from the transition Eq. 8, there are two possible roots for the steady state λ. According to Lemma 1 and Fig. 1, one of these two roots must be the desired root. We exclude the root

$$\lambda = \frac{n_u (1+q) - pn_s + \sqrt{\Delta}}{2(n_u - n_s)}$$

in several steps as follows. First, note that $n_u(1+q) - pn_s = (n_u - n_s)p + n_u(1-p+q) > 0$.

Then, rewrite $\Delta = \left[n_u (1+q) - pn_s\right]^2 - 4(n_u - n_s) n_u q$ as

$$\begin{aligned}
\Delta &= \left[(n_u - n_s) p - n_u (1-p+q)\right]^2 + 4p(1-p+q) n_u (n_u - n_s) - 4q (n_u - n_s) n_u \\
&= \left[(n_u - n_s) p - n_u (1-p+q)\right]^2 + 4n_u (n_u - n_s) \left[p(1-p+q) - q\right] \\
&= \left[(n_u - n_s) p - n_u (1-p+q)\right]^2 + 4n_u (n_u - n_s)(1-p)(p-q) > 0.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\lambda &> \frac{(n_u - n_s) p + n_u (1-p+q) + \left[(n_u - n_s) p - n_u (1-p+q)\right]}{2(n_u - n_s)} \\
&= \frac{2(n_u - n_s) p}{2(n_u - n_s)} > p.
\end{aligned}$$

This contradicts $q < \lambda^* < p$ in Lemma 1. □

When government education spending is optimally chosen, it is a function of λ_t in Proposition 9. The analysis of the steady state and stability for this case is more complex and relegated to Electronic supplementary material—Appendix B. It shows that, when government education spending is optimally chosen, there may be more than one steady state at different levels. Recall that the exogenous government spending assures a unique and stable steady state, which is essential for comparative statics in this section. Thus, we assume exogenous government spending in this section.

Let us now look at how the steady-state ratio of skilled to working population is associated with fertility rates and the probabilities for children to become skilled.

Lemma 5 *At the steady state, we have*

$$\frac{\partial\lambda}{\partial n_s} > 0,\ \frac{\partial\lambda}{\partial n_u} < 0,\ \frac{\partial\lambda}{\partial p} > 0,\ \frac{\partial\lambda}{\partial q} > 0.$$

Proof We use $q < \lambda < p$ in the steady state given in Lemma 1 and Lemma 4 to sign these derivatives in Lemma 5. Differentiating Eq. 8 in the steady state leads to

$$\frac{\partial\lambda}{\partial n_s} = \frac{(p-\lambda)\,\lambda}{-2\,(n_u - n_s)\,\lambda + (1+q)\,n_u - pn_s},$$

$$\frac{\partial\lambda}{\partial n_u} = \frac{(q-\lambda)\,(1-\lambda)}{-2\,(n_u - n_s)\,\lambda + (1+q)\,n_u - pn_s},$$

$$\frac{\partial\lambda}{\partial p} = \frac{\lambda n_s}{-2\,(n_u - n_s)\,\lambda + (1+q)\,n_u - pn_s},$$

$$\frac{\partial\lambda}{\partial n_s} = \frac{(1-\lambda)\,n_u}{-2\,(n_u - n_s)\,\lambda + (1+q)\,n_u - pn_s}.$$

According to Lemma 4, the common denominator of these derivatives is just $\sqrt{\Delta} > 0$. Therefore, the signs of these derivatives are as claimed. □

In the steady state, we know that the average potential income is

$$\lambda w_s + (1-\lambda)\,w_u = \lambda\,(w_s - w_u) + w_u.$$

Thus, the greater is λ, the higher is the average income.

Let the steady-state value of the ratio of skilled to working population under private education be denoted by λ^r. Then, according to Lemma 5 under private education, clearly we have the following observation.

Proposition 12 *In the steady state under private education, the fraction of the working population as skilled workers is increasing in the wage rate of skilled and unskilled workers, w_s and w_u, as well as in the productivity parameter in the education technology, γ_s and γ_u*

$$\frac{\partial\lambda^r}{\partial w_i} > 0\,(i = s \text{ or } u)\,;$$

$$\frac{\partial\lambda^r}{\partial \gamma_i} > 0\,(i = s \text{ or } u)\,.$$

These results are intuitive. When the wages of all workers rise, all children's education will benefit from more education spending and hence more children can become skilled. When the education sector becomes more effective for all children, the chances of children becoming skilled will be enhanced, given any education inputs. By increasing the steady-state ratio of skilled to working population, higher wages or higher efficiency in education tend to reduce (raise) the relative number of unskilled (skilled) parents whose children become skilled (unskilled). On the other hand, higher wages or higher efficiency in education for skilled parents tend to increase p and tend to reduce the relative number of skilled parents whose children experience downward mobility. On the other hand, higher wages or higher efficiency in education for unskilled parents tend to increase q and tend to increase the relative number of unskilled parents whose children experience upward mobility.

Let the steady-state value of skilled to working population under public education be denoted by λ^{g}. Then, paralleling Proposition 12, we have the following proposition with exogenous government spending.

Proposition 13 *In the steady state under public education with fixed government education spending, the fraction of the working population as skilled workers is independent of the wage rates of skilled and unskilled workers, w_s and w_u, and increasing in the productivity parameter in the education technology, γ_s and γ_u, as well as in the exogenous government spending:*

$$\frac{\partial \lambda^{g}}{\partial w_i} = 0\,(i = s \text{ or } u)\,;$$
$$\frac{\partial \lambda^{g}}{\partial \gamma_i} > 0\,(i = s \text{ or } u)\,;$$
$$\frac{\partial \lambda^{g}}{\partial g} > 0.$$

By increasing the steady-state ratio of skilled to working population, higher education efficiency and government education spending tend to reduce (increase) the relative number of unskilled (skilled) parents experiencing upward (downward) mobility under public education. Higher wages have no effect on intergenerational mobility for fixed government spending.

From the last part of Proposition 13 and Lemma 5, we obtain the following result:

Proposition 14 *There exists a unique level of fixed government education spending g^*, such that average potential income in the steady state will be higher in the public than in the private education regime if and only if $g > g^*$.*

Proof Given $(w_{\mathrm{s}}, w_{\mathrm{u}})$, a higher average potential income in the steady state means a higher ratio of skilled to working population λ in the steady state. Define $\phi^i \equiv n_{\mathrm{u}}^i / n_{\mathrm{s}}^i$, where $i = r, g$ indicates the private or public education regime. Rewrite the steady-state ratio of skilled to working population as

$$\lambda^i = \frac{\phi^i \left(1 + q^i\right) - p^i - \sqrt{\left[\phi^i \left(1 + q^i\right) - p^i\right]^2 - 4\left(\phi^i - 1\right)\phi^i q^i}}{\phi^i - 1}.$$

From Lemma 2 and Lemma 3, it is easy to verify that $\phi^{\mathrm{r}} > \phi^{\mathrm{g}} > 1$. From Lemma 5, this lower ratio of fertility of unskilled parents to fertility of skilled parents increases the ratio of skilled to total population. In addition, the time input of unskilled parents in child education is the same in both education regimes. Thus, $q^{\mathrm{g}} > q^{\mathrm{r}}$ as long as $g > d_{\mathrm{u}}^{\mathrm{r}}$, which raises the ratio of skilled to working population according to Lemma 5. It is now clear that $p^{\mathrm{g}} = p^{\mathrm{r}}$ is sufficient (not necessary) for $\lambda^{\mathrm{g}} > \lambda^{\mathrm{r}}$. From Proposition 4, the time input of skilled parents in child education is lower in the public education regime than in the private education regime. Thus, we obtain $\lambda^{\mathrm{g}} > \lambda^{\mathrm{r}}$ if

$$g > \overline{g} = \left(k_{\mathrm{s}}^{\mathrm{r}}/k_{\mathrm{s}}^{\mathrm{u}}\right)^{\delta/(1-\delta)} d_{\mathrm{s}}^{\mathrm{r}}.$$

Combining these arguments with the last part of Proposition 13, we have $g^{*} < \overline{g}$. Recall that the tax rate to finance government education spending does not change fertility and time inputs in this model. Therefore, there are plausible parameterizations such that $0 < \tau^{*} < 1$ exists and satisfies the government budget constraint in Eq. 38 to finance g^{*}. Using the government budget constraint and solutions for the variables involved, we have a tax rate $\overline{\tau}$ corresponding to $\overline{g}$:

$$\overline{\tau} = \frac{(1+\beta)}{\beta} \frac{w_{\mathrm{s}}\left[n_{\mathrm{s}}^{\mathrm{g}}\lambda^{\mathrm{g}} + n_{\mathrm{u}}^{\mathrm{g}}\left(1-\lambda^{\mathrm{g}}\right)\right]}{\left[\lambda^{\mathrm{g}} w_{\mathrm{s}} + \left(1-\lambda^{\mathrm{g}}\right) w_{\mathrm{u}}\right]} \left(\frac{1-\alpha\delta}{1-\alpha}\right)^{\delta/(1-\delta)} \frac{\alpha\xi\,(1-\delta)}{1-\alpha}$$

which can be smaller than 1 (for budget feasibility) if the taste for child education outcome α is weak enough or if the time cost of rearing a child ξ is small enough. Thus, a tax rate at this level can yield a higher steady-state ratio of skilled to working population in the public than in the private education regime. □

This proposition implies that whether the public educational system yields a higher level of average income in the steady state depends on the amount of public expenditure on education.[11] An important feature of this model is the smaller differential in fertility between skilled and unskilled parents in the public than in the private education regime. As seen in the proof, this smaller differential in fertility leads to a higher steady-state ratio of skilled to working population in the public than in the private education regime, other things being equal. On the other hand, the smaller time input of skilled parents in child education in the public than in the private education regime may lead to a lower steady-state ratio of skilled to working population in the former than in the latter regime. If government education spending is large enough, then the steady-state ratio of skilled to working population (hence average steady-state potential income or efficiency) can be higher in the public than in the private education regime.

[11] Note that in this paper, the wage rates for skilled and unskilled are fixed. Thus, both the average output and the average utility only depend on the proportion of skilled individuals.

Under some parameter configurations (especially greater education efficiency parameters γ_s and γ_u), the optimal solution of g to social welfare maximization can be greater than g^*. In this case, public education will generate not only a higher level of social welfare but also a higher level of average income. Under some other circumstances, however, the value of g chosen by the government can be significantly lower than g^*. In this case, the private educational system will yield a higher level of efficiency in terms of average income.[12]

Comparing Propositions 13 and 14 with Proposition 6, public education spending reduces average income in the short run compared to private education but once public education spending is large enough, it can increase average income in the long run. The short-run decline in average income from public education spending (hence a short-run negative effect on growth in average income) is mainly because all parents (skilled and unskilled alike) have more children and because skilled parents spend less time for the education of each child. In particular, under public education, the probability of a child from a skilled parent to become skilled is smaller due to less investment in that child's education. The possible long-run increase in average income caused by public education spending (hence a positive effect on growth in average income in the transition toward a new steady state) works through increasing the ratio of skilled to the working population via the dynamic system captured in Eq. 8. On the one hand, this is because of the redistributive mechanism that helps the children of the poor (unskilled) to acquire education as in some existing models (see de la Croix and Doepke 2004, Cardak 2005, and the literature surveyed there). On the other hand, the differential in fertility between skilled and unskilled parents becomes smaller in the public education regime, which tends to raise the steady-state ratio of skilled to working population. Through reducing the differential in fertility, public education tends to reduce (increase) the number of parents whose children experience upward (downward) mobility.

8 Conclusion

Intergenerational mobility and income inequality are important agendas of research in both the theoretical and the empirical literature of economics.[13] While income inequality measures the equity of the current generation, intergenerational mobility is a crucial indicator of equal opportunities for future generations. In line with Becker and Lewis (1973) and more recently de la Croix and Doepke (2003, 2004), we believe that a better understanding of

[12] In this section, for simplicity, we do not conduct the analysis of social welfare, which would be qualitatively similar to that in Section 6.

[13] See, for example, Solon (2002) for a survey of the empirical literature of intergenerational mobility. Booth and Kee (2009), Lindquist and Lindquist (2011), Riphahn and Schieferdecker (2011), and Wilson et al. (2011) are some more recent empirical studies.

the observed evidence of intergenerational mobility can be obtained from the consideration of endogenous fertility, specifically differential fertility rates across rich and poor families. To this aim, this paper develops an overlapping generations model that analyzes differential fertility, education investment, inequality, and intergenerational mobility with skilled and unskilled workers in a unified framework.

The model shows that under reasonable assumptions, unskilled workers choose higher fertility but spend less time and income for a child's education than skilled workers. The switch from private education to public education raises fertility rates of all workers and reduces the relative gap between the fertility rates of unskilled and skilled workers. Also, public education raises intergenerational mobility if the wage differential is sufficiently large or if the taste for child education is sufficiently strong. The new light this paper sheds on mobility comes from the negative response of the relative fertility differential between unskilled and skilled workers to public education. This reduced relative fertility differential under public education means more children from skilled workers compared to the private education regime and tends to raise the ratio of skilled to working population in future times, other things being equal. This rise in the skilled portion of the future working population can in turn raise average income and hence raise government spending on education in future times (particularly so if the wage differential is large). Subsequently, the rise in future government spending under public education will brighten the chance for children with unskilled parents to become skilled (upward mobility) and will also mitigate the increase in the downward mobility of children with skilled parents. As a result, switching from private to public education tends to raise the downward mobility of children with skilled parents more in the short run than in the long run on the one hand and raise the upward mobility of children with unskilled parents in all times on the other hand. Also, there are very different responses of fertility differential and intergenerational mobility to a change in a preference or technology parameter under both private and public educational systems.

Moreover, the ratio of skilled to working population converges to a unique and stable steady-state level in the private education regime and likely in the public education regime with exogenous government education spending. When the ratio of skilled to working population rises in the development process, average income rises and average fertility falls; however, income inequality first rises and then falls. This pattern of movement of the key variables is broadly consistent with the observed ones in the last two centuries in developed countries. Optimal government education spending has been considered as well, even when skilled parents are allowed to opt out of public education after they pay the tax for public education.

The results of the paper may have useful policy implications. For example, governments may be concerned about the fact that better educated families often have fewer children. This paper shows that the switch from private education to public education reduces the relative gap between fertility rates of unskilled and skilled workers. Also, many developed countries have been under the pressure of low fertility, rising inequality, and falling intergenerational mobility since the 1970s (e.g., in the UK or the USA). Our analysis suggests that shifting the cost of education from the state to households may be a cause of these problems. Therefore, the government may need to design effective mechanisms to improve the efficiency of public education and increase the expenditure on public education rather than simply use private educational system as a substitute for public education.

To highlight essentials, we have used a simple stylized model in this paper. In future research, this paper can be extended in several ways. For example, in our model, the wage rates for both skilled and unskilled are constant over time. We may consider the complementary and substitution effects between skilled and unskilled workers, which may yield more realistic implications. The rising ratio of skilled to working population may push the skilled wage down and the unskilled wage up, which may induce higher fertility of skilled parents and lower fertility of unskilled parents. These substitution effects of higher wages can be canceled out by the income effect in our model with log utility such that fertility rates are independent of the wage levels. Whether these possible changes in fertility caused by the wage movements reverse falling average fertility in a different model appears to be an interesting issue for future research. Also, we may consider alternative definitions of intergenerational mobility that capture not only the odd ratio but also the portion of skilled/unskilled workers or the portion of children in each type of family. Such considerations may further enhance our understanding of intergenerational mobility in a dynamic framework. Moreover, this paper has assumed that parents are not allowed to provide private education in addition to the common level of public education spending. In future research, it will be interesting to explore the interactions between endogenous fertility, intergenerational mobility, and income inequality in a more flexible framework that considers various forms of combination between public and private educational systems.

Acknowledgements We are grateful to two anonymous referees for constructive comments and suggestions, which improved the paper significantly. The research is financed by the National University of Singapore (grant # R122000117).

References

Anger S, Heineck G (2010) Do smart parents raise smart children? The intergenerational transmission of cognitive abilities. J Popul Econ 23(3):1105–1132

Ballantine JH (2001) The sociology of education: a systematic analysis. Prentice-Hall, Upper Saddle River, NJ

Barro RJ, Sala-i-Martin X (1992) Convergence. J Polit Econ 100(2):223–251

Becker GS (1991) A treatise of the family. Harvard University Press, Cambridge, MA

Becker G, Lewis HG (1973) On the interaction between the quantity and quality of children. J Polit Econ 81(2):S279-S288

Becker G, Tomes N (1979) An equilibrium theory of the distribution of income and intergenerational mobility. J Polit Econ 87(6):1153–1189

Benabou R (1996) Heterogeneity, stratification, and growth: macroeconomic implications of community structure and school finance. Am Econ Rev 86(3):584–609

Behrman JR, Foster AD, Rosenzweig MR, Vashishtha P (1999) Women's schooling, home teaching, and economic growth. J Polit Econ 107(4):682–714
Booth AL, Kee HJ (2009) Birth order matters: the effect of family size and birth order on educational attainment. J Popul Econ 22(2):367–397
Cardak BA (2005) Education vouchers, growth, and income inequality. Macroecon Dyn 9(1):98–121
Cremer H, Pestieau P (2006) Intergenerational transfer of human capital and optimal education policy. J Public Econ Theory 8(4):529–545
Cremer H, Gahvari F, Pestieau P (2006) Pensions with endogenous and stochastic fertility. J Public Econ 90(12):2303–2321
Cremer H, Gahvari F, Pestieau P (2011) Fertility, human capital accumulation, and the pension system. J Public Econ 95(11–12):1272–1279
de la Croix D, Doepke M (2003) Inequality and growth: why differential fertility matters. Am Econ Rev 93(4):1091–1113
de la Croix D, Doepke M (2004) Public versus private education when differential fertility matters. J Dev Econ 73(2):607–629
de la Croix D, Doepke M (2009) To segregate or to integrate: education politics and democracy. Rev Econ Stud 76(2):597–628
de la Croix D, Michel P (2002) A theory of economic growth: dynamics and policy in overlapping generations. Cambridge University Press, Cambridge, UK
Davies JB, Zhang J, Zeng J (2005) Intergenerational mobility under private vs public education. Scand J Econ 107(3):399–417
Docquier F, Paddison O, Pestieau P (2007) Optimal accumulation in an endogenous growth setting with human capital. J Econ Theory 134(1):361–378
Ehrlich I, Lui FT (1991) Intergenerational trade, longevity, and economic growth. J Polit Econ 99(5):1029–1060
Epple D, Romano RE (1998) Competition between private and public schools, vouchers, and peer-group effects. Am Econ Rev 88(1):33–62
Fan CS, Stark O (2008) Looking at the "population problem" through the prism of heterogeneity: welfare and policy analyses. Int Econ Rev 49(3):799–835
Fernandez R, Rogerson R (2003) Equity and resources: an analysis of education finance systems. J Polit Econ 111(4):858–897
Galor O (2005) From stagnation to growth: unified growth theory. In: Aghion P, Durlauf SN (eds) Handbook of economic growth. Elsevier, Amsterdam, pp 171–293
Galor O, Moav O (2002) Natural selection and the origin of economic growth. Q J Econ 117(4):1133–1192
Galor O, Moav O (2004) From physical to human capital accumulation: inequality and the process of development. Rev Econ Stud 71(4):1001–1026
Galor O, Moav O (2006) Das human-capital: a theory of the demise of the class structure. Rev Econ Stud 73(1):85-117
Galor O, Tsiddon D (1996) Income distribution and growth: the Kuznets hypothesis revisited. Economica 63(250):S103–S117
Galor O, Tsiddon D (1997) Technological progress, mobility, and economic growth. Am Econ Rev 87(3):363–382
Galor O, Weil DN (2000) Population, technology, and growth: from Malthusian stagnation to the demographic transition and beyond. Am Econ Rev 90(4):806–828
Galor O, Zeira J (1993) Income distribution and macroeconomics. Rev Econ Stud 60(1):35–52
Glomm G, Ravikumar B (1992) Public versus private investment in human capital: endogenous growth and income inequality. J Polit Econ 100(4):818–834
Glomm G, Ravikumar B (2003) Public education and income inequality. Eur J Polit Econ 19(2):289–300
Grawe ND, Mulligan CB (2002) Economic interpretations of intergenerational correlations. J Econ Perspect 16(3):45–58
Hassler J, Mora JVR (2000) Intelligence, social mobility, and growth. Am Econ Rev 90(4):888–908
Hazan M, Berdugo B (2002) Child labour, fertility, and economic growth. Econ J 112(483):810–828
Iyigun MF (1999) Public education and intergenerational economic mobility. Int Econ Rev 40(3):697–710

Lindquist MJ, Lindquist GS (2011) The dynamics of child poverty in Sweden. J Popul Econ 25(4):1423–1450
Loury GC (1981) Intergenerational transfers and the distribution of earnings. Econometrica 49(4):843–867
Lynch LM (2000) Trends and consequences of investment in children. In: Waldfogel J, Danziger S (eds) Securing the future, the Ford foundation series on asset building. Russell Sage Foundation, New York, pp 19–46
Maoz YD, Moav O (1999) Intergenerational mobility and the process of development. Econ J 109(458):677–697
Moav O (2005) Cheap children and the persistence of poverty. Econ J 115(500):88–110
Mookherjee D, Napel S (2007) Intergenerational mobility and macroeconomic history dependence. J Econ Theory 137(1):49–78
Mookherjee D, Ray D (2003) Persistent inequality. Rev Econ Stud 70(2):369–393
Owen AL, Weil DN (1998) Intergenerational earnings mobility, inequality and growth. J Monet Econ 41(1):71–104
Piketty T (1995) Social mobility and redistributive politics. Q J Econ 110(3):551–584
Piketty T (2000) Theories of persistent inequality and intergenerational mobility. In: Atkinson AB, Bourguignon F (eds) Handbook of income distribution, vol 1. Elsevier, Amsterdam, pp 429–476
Ramey G, Ramey VA (2010) The rug rat race. Brookings Pap Econ Act 41(1):129–176
Razin A, Sadka E (1995) Population economics. MIT Press, Cambridge, MA
Riphahn RT, Schieferdecker F (2011) The transition to tertiary education and parental background over time. J Popul Econ 25(2):635–675
Solon G (2002) Cross-country determinants in intergenerational earnings mobility. J Econ Perspect 16(3):59–66
Varvarigos D, Zakaria IZ (2012) Endogenous fertility in a growth model with public and private health expenditures. J Popul Econ. doi:10.1007/s00148-012-0412-1
Weinberg BA (2001) An incentive model of the effect of parental income on children. J Polit Econ 109(2):266–280
Wilson D, Burgess S, Briggs A (2011) The dynamics of school attainment of England's ethnic minorities. J Popul Econ 24(2):681–700
Zhang J (1996) Optimal public investments in education, and endogenous growth. Scand J Econ 98(3):387–404
Zhang J, Zhang J, Lee R (2001) Mortality decline and long-run economic growth. J Public Econ 80(3):485–507